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PREDICTION OF STATISTICAL SYSTEM PERFORMANCE FROM PARAMETER DISTRIBUTIONS

by

Alvin LaVerne Franson

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THESIS

PREDICTION OF STATISTICAL
SYSTEM PERFORMANCE FROM
PARAMETER DISTRIBUTIONS

by

Alvin LaVerne Franson

June 1969

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Prediction of Statistical System Performance from Parameter Distributions

by

Alvin LaVerne Franson
Lieutenant, United States Navy
B.S.E.E., University of Kansas, 1963

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

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ABSTRACT

Techniques for extrapolating statistical data on the parameters of individual components to the statistical performance index for an overall system are considered. Two cases are evaluated. (1) The deterministic case in which the system's performance index is known functionally in terms of the system parameters. (2) The non-deterministic case in which only limited data on the performance index and its sensitivity with respect to system parameters is known. Computer programs are developed in both cases for combining given probability density distributions for the parameters into an overall probability density distribution for the system performance index. Theory and programs are developed and verified with a specific numerical example.

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I. INTRODUCTION

The performance of a system, from a statistical point of view, depends upon the nominal values of the parameters of the component parts included in the system and on the statistical distributions of those parameters. Alternately, the response of the component parts to a given signal may be known statistically and it is desired to predict an overall statistical distribution for the system response. Statistical information on the system's response may be interpreted in terms of probability of success or failure (operation or non-operation) for a single system or in terms of the percentage of operational systems that can be expected in an ensemble of systems. Currently three basic methods are being used to obtain performance data for systems; the worst-case method [Mark and Stember 1965], the moment method [Mark and Stember 1965], and the Monte Carlo method [Shreider 1966].

The worst-case method is a non-statistical approach intended to determine whether it is possible, with given parameter tolerance limits, for the systems performance characteristics to fall outside specifications. This method determines nothing of the probability distribution of the performance. Worst-case analysis, although it may be involved, does provide useful engineering insight and is used a great deal in practice. However, its outlook is basically limited.

Statistics are combined with system analysis techniques in the moment method to estimate the probability that performance will remain within specified limits. The actual parameter probability distributions are usually replaced by Gaussian distributions having the same variances. Hence, the moment method has inherent limitations. Parameter probability distribution characteristics such as skewness and peakedness are usually not taken into account.

In the Monte Carlo method a large number of alternate replicas of a system are simulated by mathematical models. Component values are randomly selected and the performance of each replica is determined for its particular set of components. The accuracy of the Monte Carlo method is dependent upon taking a sufficiently large number of replicas and can be both lengthy and costly. Monte Carlo methods are also used extensively in practice although they provide limited engineering information.

The purpose of this thesis is to present a numerical technique for combining statistical data on several independent components into a single distribution function for the system performance. This is accomplished by considering two general cases. The deterministic case, where the functional relationships between the system parameters and the output are known functionally; and the non-deterministic case, where these functional relationships are unknown but measured discrete data is available. In both cases the actual parameter probability distributions are used in conjunction with numerical integration techniques to obtain the system distribution function. One major advantage of the approach is that the known parameter distributions are quite arbitrary and may be histograms obtained from the results of experiments on individual components.

II. DETERMINISTIC CASE

The object of the deterministic case is to investigate the distribution of a system output when the output has a known mathematical relationship to several internal parameters. The basic assumptions are that all of the internal parameters are independent and that the probability distributions of the internal parameters, although arbitrary, are known mathematically or given in terms of histograms.

To commence the investigation, the general equations for various combinations of parameter distributions which may be expected in a typical system performance index must be obtained. These will basically consist of operations on the distributions of random variables which are interrelated by addition, subtraction, multiplication, and division. The basic mathematics for these operations are well known in the literature, at least in general form [Beckmann 1967].

Consider two independent random variables: X, assuming values x with a known probability density $p_X(x)$, and Y, assuming values y with a known probability density $p_Y(y)$. Also consider the functions $U = U(X, Y)$ and $V = V(X, Y)$ and their inverse functions $X = X(U, V)$ and $Y = Y(U, V)$. Then, when X is near x and Y is near y, U and V must be near $u = u(x, y)$ and $v = v(x, y)$. Hence,

$$p_{XY}(x, y)\, dxdy = p_{UV}(u, v)\, dudv \qquad (1)$$

Substituting the inverse functions and the required probability densities, we obtain

$$p_{UV}(u, v) = p_X\,[x(u, v)]\; p_Y\,[y(u, v)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \qquad (2)$$

where $p_{UV}(u, v)$ is the joint probability density of U and V, $p_X[x(u, v)]$ is the probability density of X, $p_Y\,[y(u, v)]$ is the probability density of Y, $x(u, v)$ and $y(u, v)$ are the inverse functions, and $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ is the Jacobian, given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad (3)$$

When X and Y are related by a single functional the foregoing can be modified by letting $U = U(X, Y)$ and $V = X$. (4)

Then,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & 1 \\ \frac{\partial y}{\partial u} & 0 \end{vmatrix} = - \frac{\partial y}{\partial u} \qquad (5)$$

Equation (2) then becomes

$$p_{UX}(u, x) = p_X(x) p_Y\,[y(u, x)] \left| \frac{\partial y}{\partial u} \right| \qquad (6)$$

Since, in general, $p_U(u) = \int_{-\infty}^{\infty} p_{UX}(u, x)dx$, equation (6) may be written as

$$p_U(u) = \int_{-\infty}^{\infty} p_X(x) p_Y\,[y(u, x)] \left| \frac{\partial y}{\partial u} \right| dx \qquad (7)$$

Using equation (7), the specific formulas of Table I have been compiled for the functional relationships and operations listed, which were chosen on the basis of what might be expected in engineering problems.

The technique outlined here consists of the successive application of the formulas of Table I as required, with the required integration performed numerically. Various types of numerical integration are possible; rectangular integration, trapezoidal integration, or Simpson's Rule. These are summarized as follows:

Rectangular integration [Stanton 1961],

$$\int_{X_1}^{X_n} f(x)\,dx = \Delta x \sum_{i=1}^{n} f(x_i) \tag{8}$$

Trapezoidal integration [Stiefel 1963],

$$\int_{X_1}^{X_n} f(x)\,dx = \Delta x \left\{ \frac{1}{2}\Big[f(x_1) + f(x_n) \Big] + \sum_{i=2}^{n-1} f(x_i) \right\} \tag{9}$$

Simpson's Rule [Stanton 1961],

$$\int_{X_0}^{X_0+n\Delta X} f(x)\,dx = \frac{\Delta x}{3} \sum_{i=0}^{n} C f(x_i) \tag{10}$$

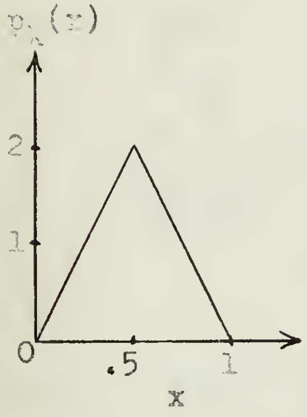
where C = 1, 4, 2, --- 2, 4, 1

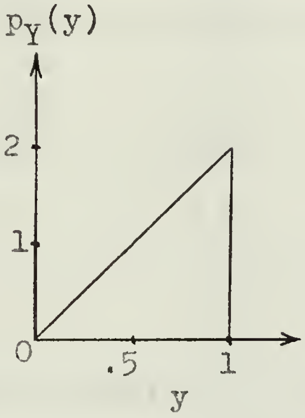
Although the specific technique of numerical integration is of prime importance in so far as accuracy and computer programing and time are concerned, the exact procedure used is a detail which is best dictated by the particular problem to which these techniques are to be applied.

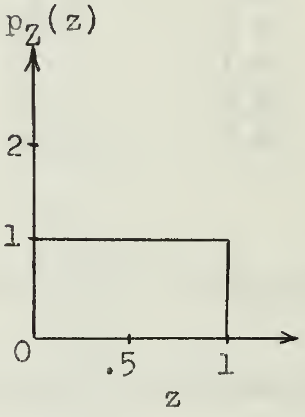
This method may be best illustrated by considering the following example. Consider that:

$$H = \frac{\text{[illegible]}}{XY} + h + Z \tag{11}$$

where a and b are constants and X, Y, and Z are independent random variables with known (measured) probability densities. Consider that $p_X(x)$, $p_Y(y)$, and $p_Z(z)$ are given as follows:







(a) Consider first the combination U = XY, with $\Delta x = 0.1$. Using rectangular integration for simplicity, equation (3) of Table I becomes, for $u = u_1$:

$$p_U(u_1) = \int_{-\infty}^{\infty} p_X(x) p_Y(u/x) \frac{dx}{|x|} \cong \Delta x \sum_{i=1}^{9} \frac{p_X(x_i)}{x_i} p_Y\left(\frac{u_i}{x_i}\right) \qquad (12)$$

When $u = u_1 = 0.09$

x_i	u_1/x_i	$\frac{p_X(x_i)}{x_i}$	$p_Y(u_1/x_i)$	$\frac{p_X(x_i)}{x_i} p_Y(u_1/x_i)$
0.1	0.9	4.0	1.8	7.2
0.2	0.45	4.0	0.9	3.6
0.3	0.3	4.0	0.6	2.4
0.4	0.22	4.0	0.43	1.72
0.5	0.18	4.0	0.37	1.48
0.6	0.15	2.6	0.30	0.78
0.7	0.13	1.7	0.26	0.44
0.8	0.11	1.0	0.21	0.21
0.9	0.1	0.44	0.20	0.088
			sum =	17.918

$p_U(u_1) = (0.1)(17.918) = 1.79$

when $u = u_2 = 0.2$

x_i	u_2/x_i	$\frac{p_X(x_i)}{x_i}$	$p_Y(u_2/x_i)$	$\frac{p_X(x_i)}{x_i}\, p_Y(u_2/x_i)$
0.1	2.0	4.0	0.0	0.0
0.2	1.0	4.0	2.0	8.0
0.3	0.7	4.0	1.4	5.6
0.4	0.5	4.0	1.0	4.0
0.5	0.4	4.0	0.8	3.2
0.6	0.33	2.6	0.67	1.74
0.7	0.3	1.7	0.6	1.02
0.8	0.25	1.0	0.5	0.5
0.9	0.22	0.44	0.42	0.185
			sum =	24.245

$p_U(u_2) = (0.1)(24.245) = 2.4$

The pattern for evaluating $p_U(u)$ is repeated for $u = .3, .4, \text{--}, 1.0$ yielding the curve of Figure 1.

(b) Next consider the function $V = \frac{1}{U} = \frac{1}{XY}$. Here equation (7) of Table I may be applied.

$$p_V(v) = \frac{1}{v^2} p_U(1/v) = u^2 p_U(u) \qquad (13)$$

u	v	$p_U(u)$	u^2	$p_V(v)$
0.1	10.0	1.8	0.01	0.018
0.2	5.0	2.42	0.04	0.097
0.3	3.33	2.39	0.09	0.215
0.4	2.5	2.11	0.16	0.338
0.5	2.0	1.54	0.25	0.385
0.6	1.66	1.02	0.36	0.364
0.7	1.43	0.55	0.49	0.270
0.8	1.25	0.28	0.64	0.179
0.9	1.11	0.05	0.81	0.040

$p_V(v)$ is plotted in Figure 2.

(c) The function W may now be calculated using equation (8) of Table I, where $W = aV + b = \frac{a}{XY} + b$ (14)

and

$$p_w(w) = \frac{1}{|a|} p_V\left(\frac{w-b}{a}\right) = \frac{1}{|a|} p_V(v) \tag{15}$$

If a = 2 and b = 1, the following result is obtained.

V	W = aV + b	$p_V(v)$	$p_V(v)/2$
1	3	0.0	0.0
2	5	0.385	0.1925
3	7	0.25	0.125
4	9	0.15	0.075
5	11	0.10	0.050
6	13	0.06	0.030
7	15	0.04	0.020
8	17	0.03	0.015
9	19	0.02	0.010
10	21	0.01	0.005

The result is plotted in Figure 3.

(d) Since, from equations (11) and (14),

$$H = W + Z \tag{16}$$

it follows from equation (1) of Table I that $p_H(h)$ is given by the convolution of $p_W(w)$ and $p_Z(z)$. This operation may also be carried out numerically and for the given $p_Z(z)$ may be expressed as

$$\left.\begin{aligned} p_H(h) &= \int_0^h p_W(w)\,dw \quad \text{for} \quad 0 \leq h \leq 1 \\ p_H(h) &= \int_{h-1}^h p_W(w)\,dw \quad \text{for} \quad h > 1 \end{aligned}\right\} \tag{17}$$

These integrals may also be carried out numerically yielding the results plotted in Figure 4.

To work more complicated problems than the previous example requires a great amount of mathematical calculations. Thus, when using numerical integration techniques the method lends itself nicely to the digital computer. Figures 5 through 9 are the computer flow graphs corresponding to some of the equations of Table I. The computer program for any mathematical function can then be derived from the appropriate combinations of these basic flow graphs.

To illustrate a more complex problem consider the following example·

given

$$\theta = 1285 \left(\frac{1}{A} - \frac{1}{B} \right) + W \tag{18}$$

where

$$A = B_{3a}B_{6a} , \qquad B = B_{3b}B_{6b}$$

and

$$B_{3a} = \frac{B_{3b}}{1 + B_{3b}(2.16 \times 10^{-2})}$$

$$B_{6a} = \frac{B_{6b}}{1 + B_{6b}(0.63 \times 10^{-2})}$$

B_{3b} is a normal distribution; mean = 130, standard deviation = 10

B_{6b} is a normal distribution; mean = 65, standard deviation = 12

W is a uniform distribution from $w = -4/7$ to $w = +4/7$

Theta is system output parameter and the variables B_{3b}, B_{6b}, and W are independent.

Since the variables are independent, equation (18) can first be simplified by ordinary mathematical manipulation.

Let $X = B_{3b}$, $Y = B_{6b}$, $k_1 = 2.16 \times 10^{-2}$, $k_2 = 0.63 \times 10^{-2}$

then $B = XY$

$$A = \frac{XY}{(1 + k_1 X)(1 + k_2 Y)}$$

$$\frac{1}{A} - \frac{1}{B} = k_1 k_2 + \frac{k_1}{Y} + \frac{k_2}{X}$$

and $$\theta = 0.175 + \frac{27.8}{Y} + \frac{8.1}{X} + W \qquad (19)$$

Now, utilizing the basic methods and flow graphs, the problem can be programed on the digital computer and the probability density of θ readily obtained. The program and solution are contained in Appendix I.

III. NON-DETERMINISTIC CASE

In many practical problems the functional relationship between variables is unknown and it cannot be expressed simply by formula. It is generally available only as numerical data from a computer run on a mathematical model of the system under study. Two cases are considered here. First, the case where the system response of interest, U(X), and its sensitivity coefficients, U'(X), are available at discrete values of a single variable, X.

$$U(X) = \left[u(x_1),\ u(x_2),\ ---,\ u(x_n)\right] \tag{20}$$

$$U'(X) = \left[u'(x_1),\ u'(x_2),\ ---,\ u'(x_n)\right] \tag{21}$$

Second, the case where the system response of interest, U(X, Y), and its sensitivity coefficients, U'(X, Y), are available at discrete values of two independent variables, X and Y.

$$U(X,Y) = \begin{vmatrix} u_{11} & u_{12} & --- & u_{1n} \\ u_{21} & u_{22} & --- & u_{2n} \\ - & - & & - \\ - & - & & - \\ u_{m1} & u_{m2} & --- & u_{mn} \end{vmatrix} \tag{22}$$

where

$$u_{ij} = u(x_i, y_j)$$

and

$$U'(X, Y) = \begin{vmatrix} u'_{11} & u'_{12} & --- & u'_{1n} \\ u'_{21} & u'_{22} & --- & u'_{2n} \\ - & - & & - \\ - & - & & - \\ u'_{m1} & u'_{m2} & --- & u'_{mn} \end{vmatrix} \tag{23}$$

where

$$u'_{ij} = \frac{\partial u(x_i, y_j)}{\partial x_i}$$

In both cases the problem reduces to determining $p_U(u)$ from the known distributions $p_X(x)$, $p_Y(y)$ and the data of equations (20) and (21), or (22) and (23).

In the first case, consider that the probability density, $p_X(x)$, and the cumulative distribution,

$$P_X(x_j) = \int_{-\infty}^{x_j} p_X(x)\,dx \tag{24}$$

are known. Then at each data point of u(x),

$$P_U\left[u(x_j)\right] = P_X(x_j) \tag{25}$$

That is, the probability that $u \leq u(x_j)$ is equal the probability that $x \leq x_j$. Equation (25) gives discrete points on the cumulative distribution, $P_U(u)$, for each value of the parameter x for which u(x) has been measured.

It is also well known [Beckman 1967] that

$$p_U(u) = \frac{\Delta P_U(u)}{\Delta u} = \frac{p_X(x)}{\left|\frac{du}{dx}\right|} \tag{26}$$

Thus the slope of the cumulative distribution curve, $\frac{\partial P_j(u_j)}{\partial u_j}$, at each data point is equal to the known probability density, $p_X(x_j)$, divided by the measured sensitivity function, $\left|\frac{\partial u}{\partial x}\right|_j$, at the data point. The conclusion is that the cumulative distribution, $P_U(u)$, and its slope may be calculated readily from the measured response and sensitivity data. This curve is therefore evaluated as a piecewise linear approximation.

When U is a function of two variables, X and Y, the foregoing cumulative distribution curve can be evaluated for each value of the second variable, Y. Thus a family of such curves or data points denoted by $P_U(u, y_j)$ is available. For a specific value of $u = u_i$ each of these curves gives the probability that $u \leq u_i$ when $y = y_j$. The total probability that $u \leq u_i$ depends upon $p_Y(y)\Delta y$ and is given by the equation:

$$P_U(u_i) = \Delta y \sum_{j=1}^{m} P_U(u_i, y_j) p_Y(y_j) \tag{27}$$

where $P_U(u, y_j)$ is the cumulative distribution evaluated with y as a constant and x as a single variable as discussed previously.

This result can be proven by considering the following (see equation (7)):

$$p_U(u) = \int_{-\infty}^{\infty} p_Y(y) p_X\left[x(u, y)\right] \left|\frac{\partial x}{\partial u}\right| dy \tag{28}$$

where $u = u(x, y)$. $\frac{\partial u}{\partial x}$ is the sensitivity function with respect to the parameter x and in equation (28) must be expressed in terms of u and y. It is convenient to denote

$$\frac{\partial u}{\partial x} = u_X(u,y) \tag{29}$$

and then equation (28) becomes

$$p_U(u) = \int_{-\infty}^{\infty} p_Y(y)\, p_X\left[x(u,y)\right] \frac{dy}{u_X(u,y)} \tag{30}$$

We also know

$$P_U(u) = \int_{-\infty}^{u} p_U(u)\,du \tag{31}$$

Substituting equation (30) into equation (31) and inter-changing the order of integration yields

$$P_U(u) = \int_{-\infty}^{\infty} p_Y(y)\,dy \int_{-\infty}^{u} p_X\left[x(u,y)\right] \frac{du}{u_X(u,y)} \tag{32}$$

Now

$$\int_{-\infty}^{u} p_X\left[x(u,y)\right] \frac{du}{u_X(u,y)} = \int_{-\infty}^{u} p_U(u,y)\,du = P_{UY}(u,y) \tag{33}$$

Since, if Y is considered a constant and X as the random variable, u = u(x, y),

$$p_U(u,y) = \frac{p_X\ [x(x,y)]}{\frac{\partial u}{\partial x}} = \frac{p_X\ [x(x,y)]}{u_X(u,y)} \tag{34}$$

Substituting equation (33) into equation (32) yields:

$$P_U(u) = \int_{-\infty}^{\infty} p_Y(y) P_U(u,y)\,dy \tag{35}$$

In discrete form, equation (35) becomes equation (27) when rectangular integration is used. However, more accurate integration techniques may also be applied to equation (35).

To demonstrate the application of this non-deterministic method consider part of the previous example of equation (18). It is now assumed that the functional relationship between the output and the parameters is unknown. However, we do know the probability densities of the parameters x and y and have certain measured data which consists of measured values of x, y, and Ø and the corresponding sensitivities for each measured Ø. The data is presented in Table II. Following the procedure outlined, the problem can be programed on the digital computer and the probability density of Ø readily obtained. The program and solution are contained in Appendix II. The results can be compared with those obtained using the deterministic formulas. Noting, of course, that the solution to the deterministic problem, contained in the Appendix, includes the addition of the uniform distribution W(w) and the solution to the non-deterministic problem does not.

IV. CONCLUSIONS

The numerical technique presented here has distinct advantages over the worst-case, the moment, and the Monte Carlo methods. By utilizing the actual parameter probability distributions the results obtained are more accurate than those of the moment method and present a great deal more information than the worst-case method. The accuracy obtained is equivalent to that of a large number of Monte Carlo replicas. Thus the saving in time over the Monte Carlo method is obvious. In general, the numerical method presented here can obtain a more accurate solution in less time than any of the other current methods. However, certain limitations are inherent in the basic assumptions used to formulate the method. Since independent parameters are required, a great number of practical problems can not be handled by this method as it is presented here. By extending the mathematical derivations to include the correlation of parameters, the method could be used for systems in which the parameters are not independent.

Table I

Functional Formulas

(1) $U = X + Y$

$$p_U(u) = \int_{-\infty}^{\infty} p_X(x)p_Y(u-x)\,dx = \int_{-\infty}^{\infty} p_X(u-y)p_Y(y)\,dy$$

(2) $U = X - Y$

$$p_U(u) = \int_{-\infty}^{\infty} p_X(x)p_Y(x-u)\,dx = \int_{-\infty}^{\infty} p_X(u+y)p_Y(y)\,dy$$

(3) $U = XY$

$$p_U(u) = \int_{-\infty}^{\infty} p_X(x)p_Y(u/x)\frac{dx}{|x|}$$

(4) $U = X/Y$

$$p_U(u) = \frac{1}{u^2}\int_{-\infty}^{\infty} p_X(x)p_Y(x/u)|x|\,dx = \int_{-\infty}^{\infty} p_X(uy)p_Y(y)|y|\,dy$$

(5) $U = X^2$

$$p_U(u) = \frac{p_X(\sqrt{u}) + p_X(-\sqrt{u})}{2\sqrt{u}} \qquad u \geq 0$$

(6) $U = |X|$

$$p_U(u) = p_X(x) + p_X(-x) \qquad x \geq 0$$

(7) $U = a/X$

$$p_U(u) = \frac{a}{u^2}\, p_X(a/u)$$

(8) $U = aX + b$

$$p_U(u) = \frac{1}{a}\, p_X\!\left(\frac{u-b}{a}\right)$$

(a and b are constants)

Table II

Sample Data

x \ y	41	53	65	77	89
110	0.9266	0.7731	0.6761	0.6096	0.5616
120	0.9204	0.7669	0.6699	0.6034	0.5554
130	0.9152	0.7617	0.6647	0.5982	0.5502
140	0.9108	0.7573	0.6603	0.5938	0.5458
150	0.9069	0.7534	0.6564	0.5899	0.5419

θ

y	41	53	65	77	89
$\left\lvert \frac{\partial \theta}{\partial y} \right\rvert$	0.0165	0.00988	0.00658	0.00469	0.00352

x	110	120	130	140	150
$\left\lvert \frac{\partial \theta}{\partial x} \right\rvert$	0.000670	0.000561	0.000478	0.000413	0.000359

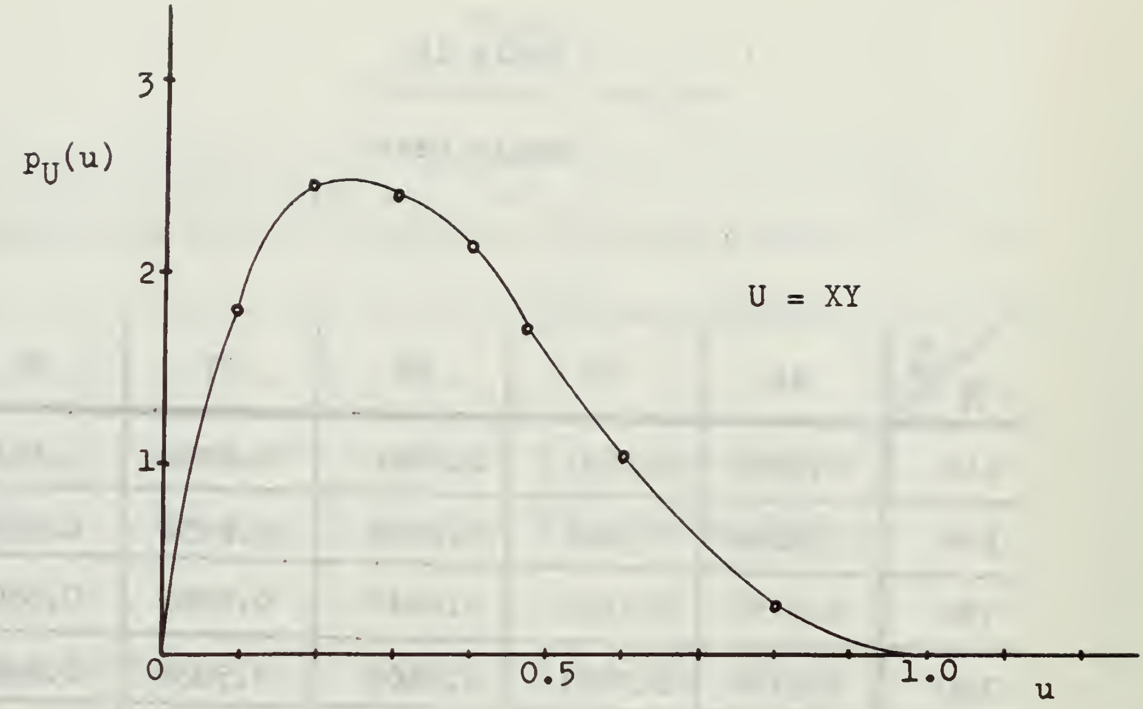


Figure 1

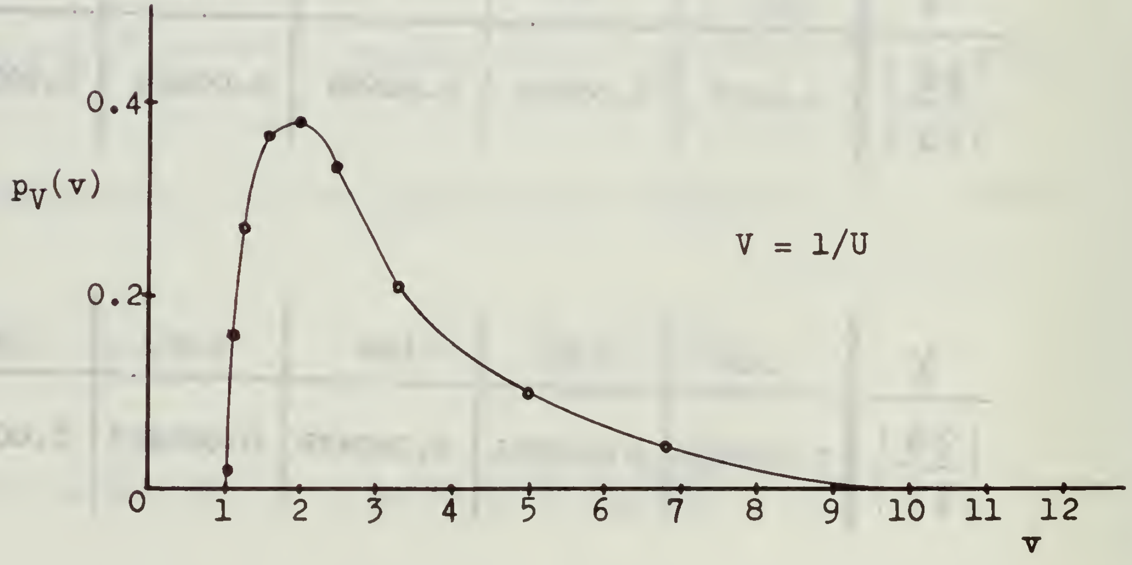


Figure 2

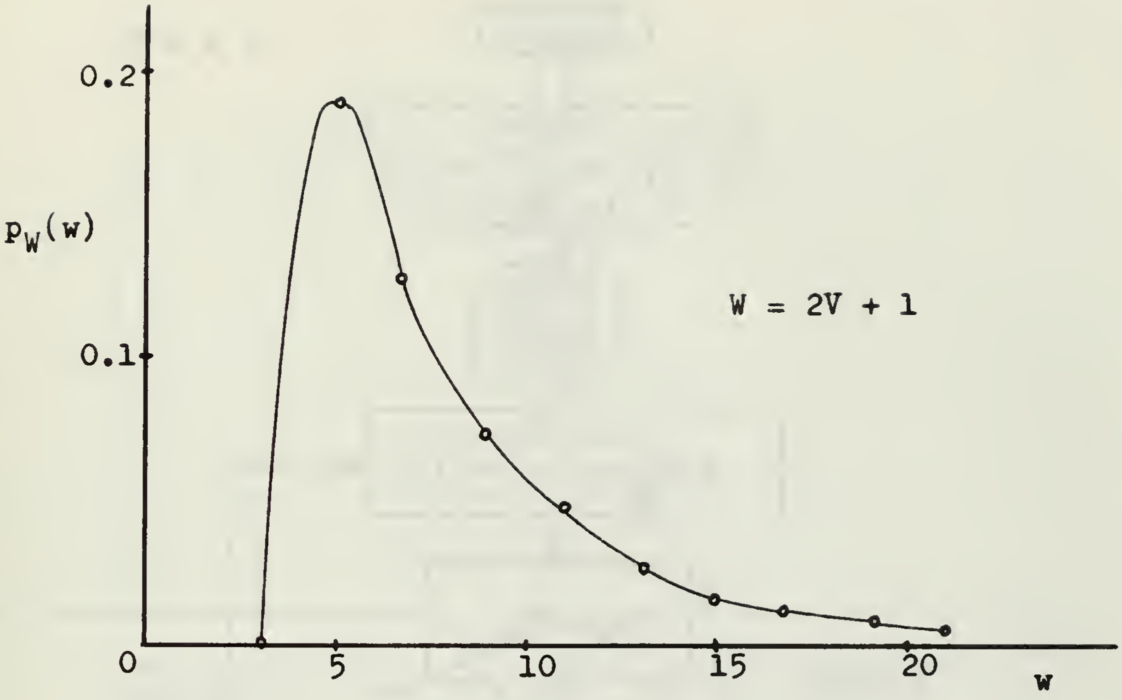


Figure 3

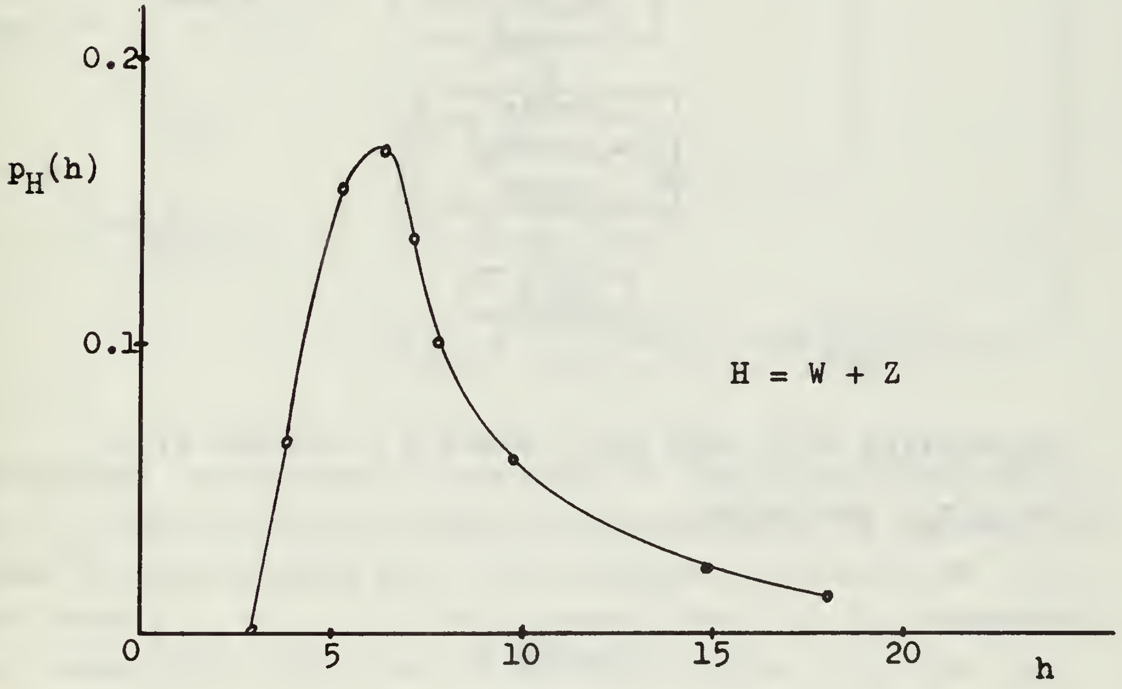
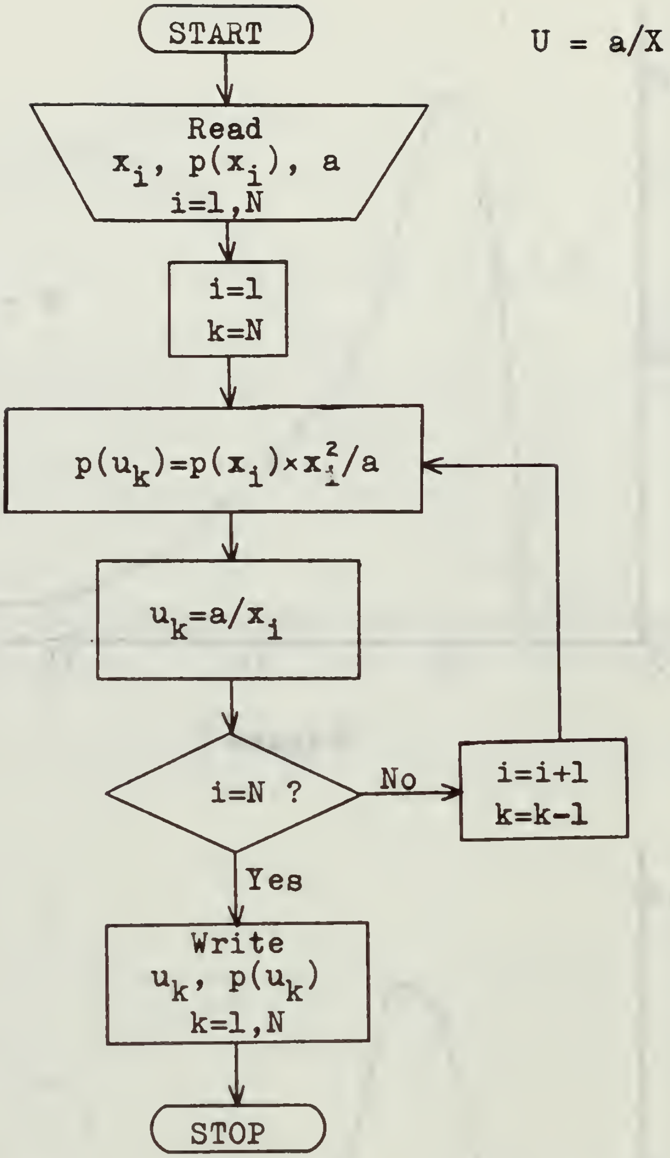
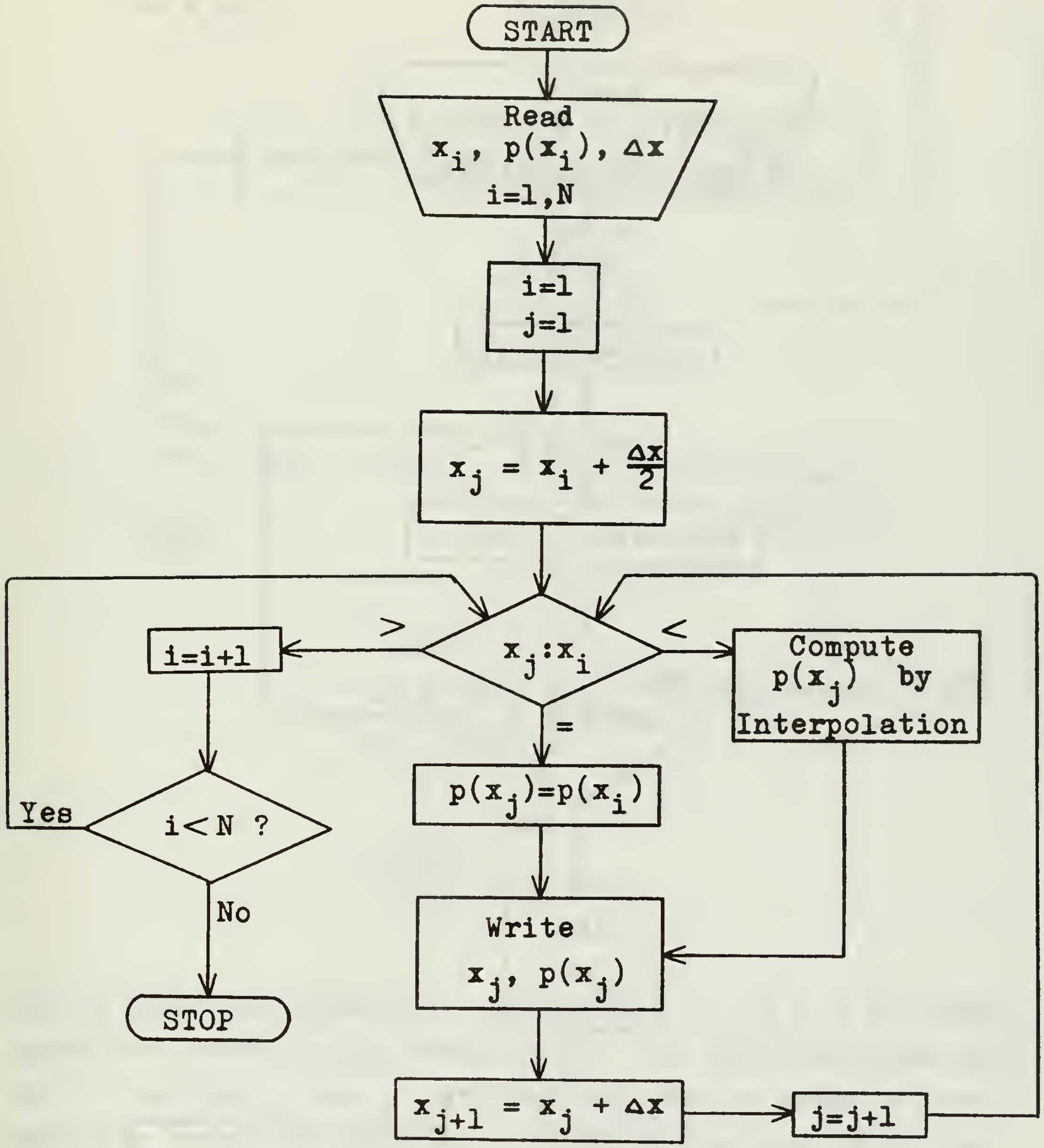


Figure 4



Calculation of u_k and $p(u_k)$ where $U = a/X$ and a, x_i, and $p(x_i)$ are known.

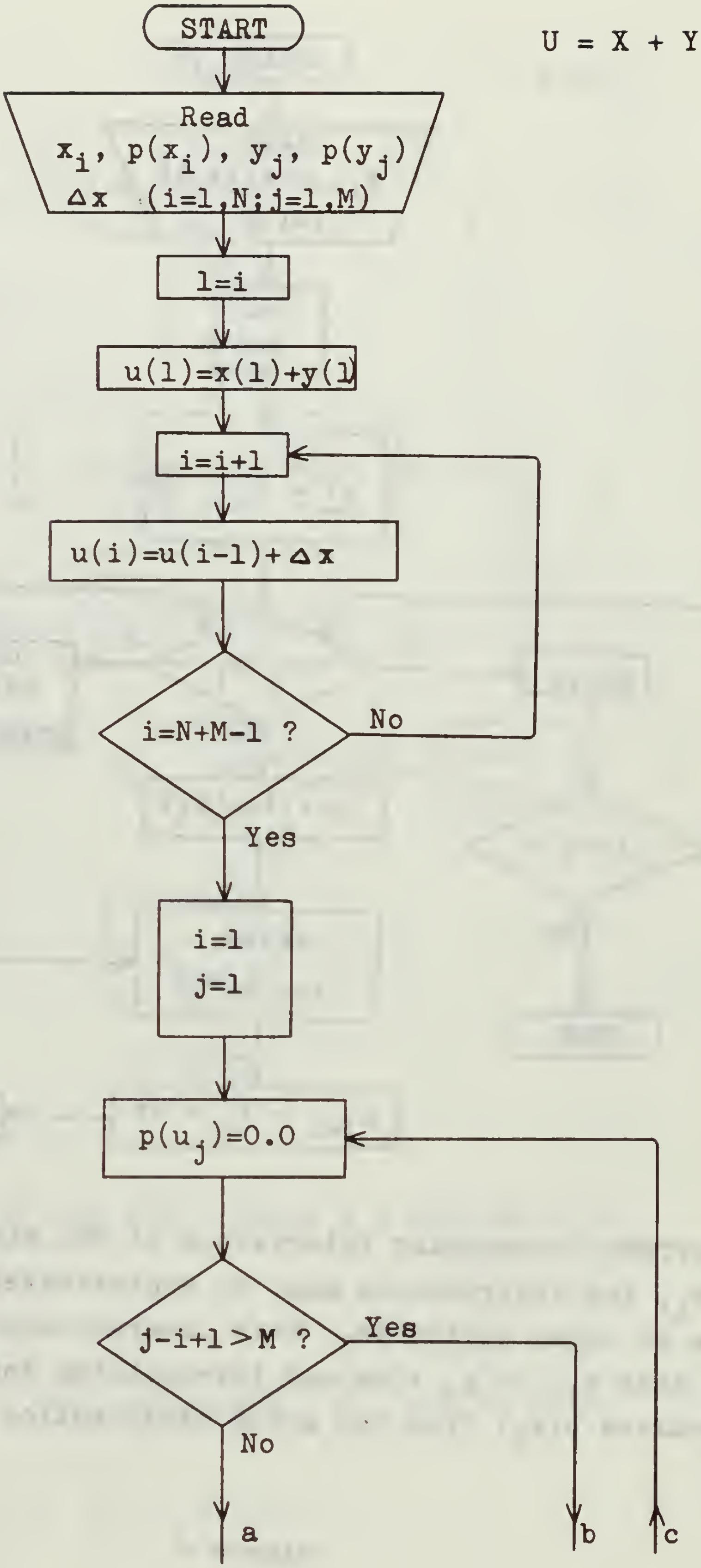
Figure 5

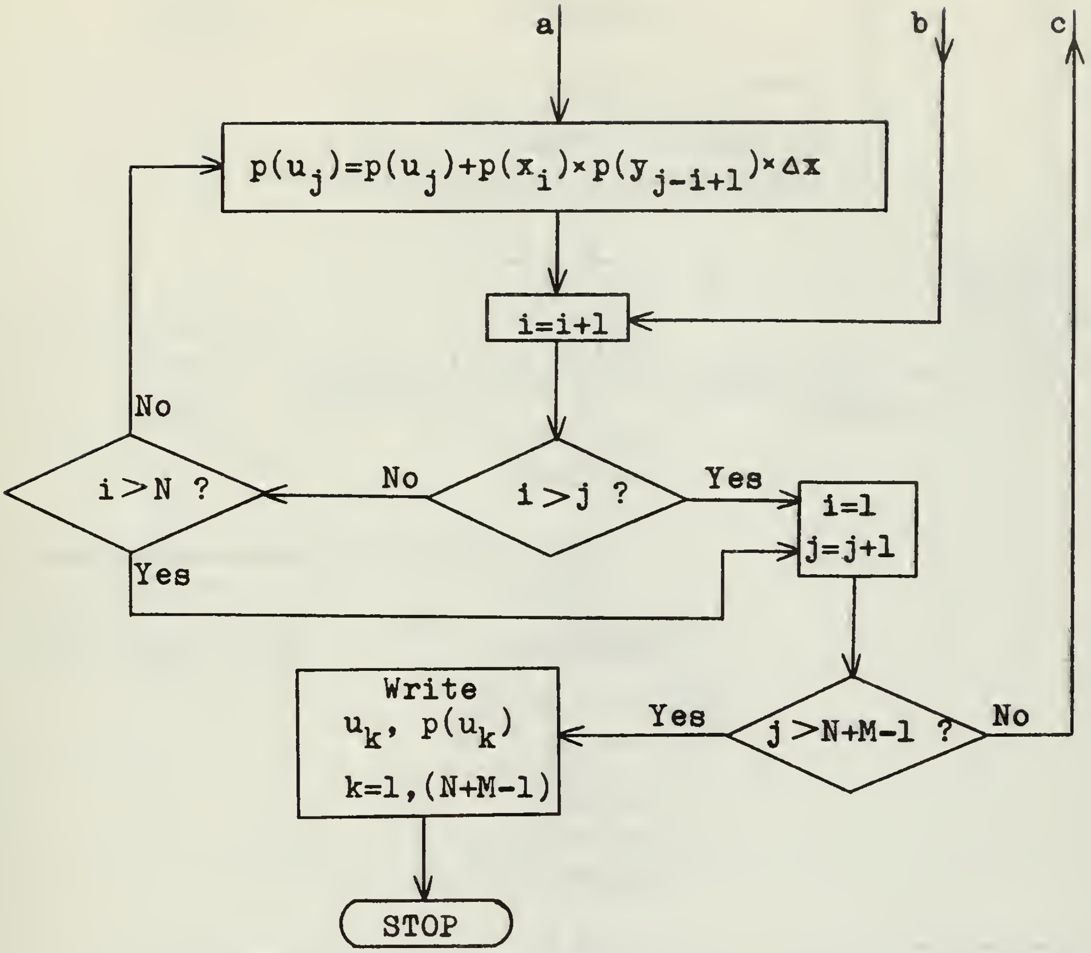


To perform rectangular integration of the distribution $p(x_i)$ vs. x_i, the distribution must be approximated by rectangular areas of equal widths Δx. This program calculates the x_j's such that $x_{j+1} = x_j + \Delta x$ and interpolates for the corresponding values $p(x_j)$ from the given distribution $p(x_i)$ vs. x_i.

Figure 6

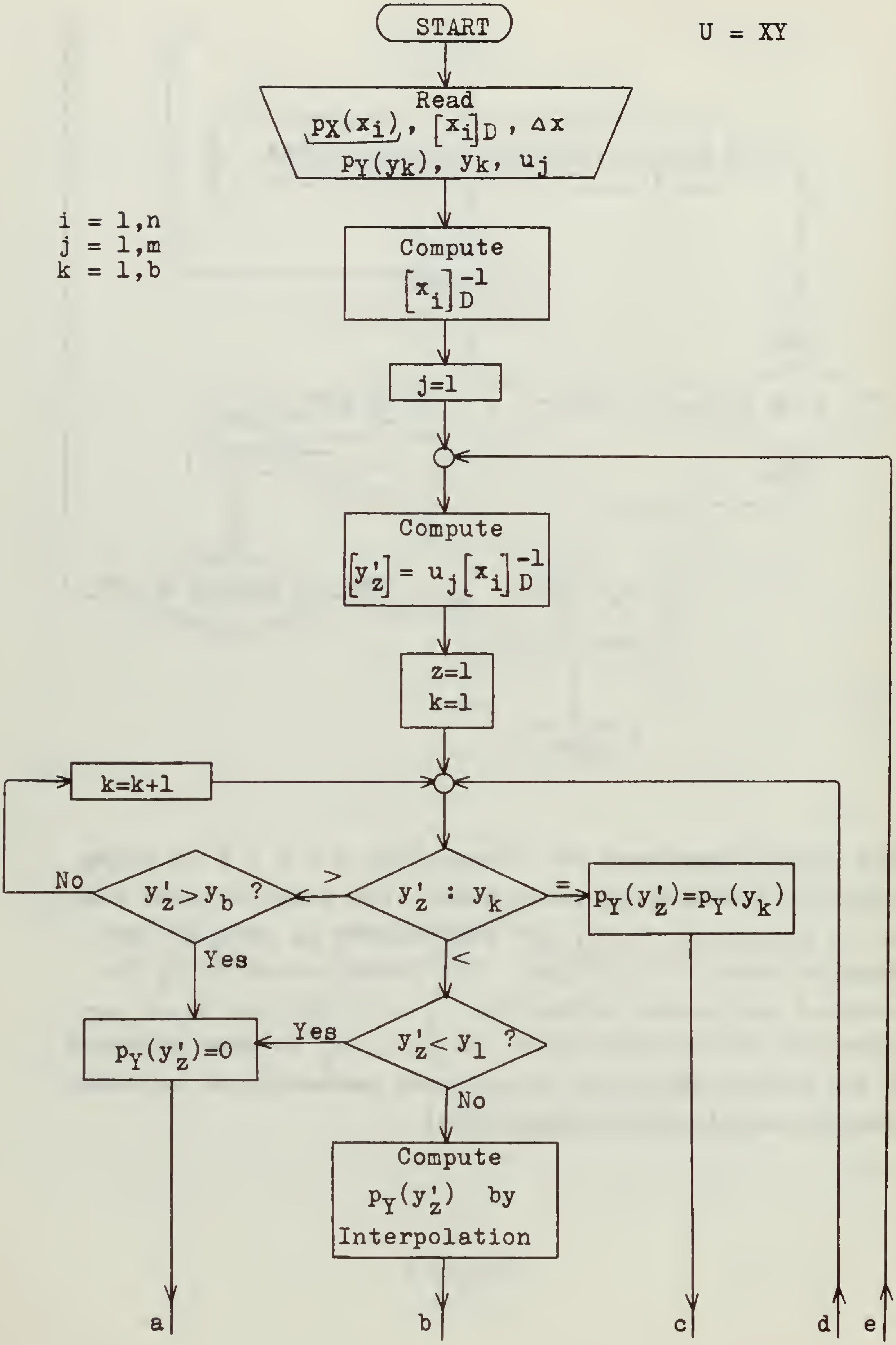
$U = X + Y$


START
Read
x_i, p(x_i), y_j, p(y_j)
Δx (i=1,N; j=1,M)
l=i
u(l)=x(l)+y(l)
i=i+l
u(i)=u(i-l)+Δx
i=N+M-l ?
No
Yes
i=l
j=l
p(u_j)=0.0
j-i+l > M ?
Yes
No
a
b
c



This program performs the convolution U = X + Y by using graphical convolution techniques. The distributions $p(x_i)$ vs. x_i and $p(y_j)$ vs. y_j are represented by sets of rectangular areas of width Δx. The distribution of x_i is reversed and passed across the y_j distribution in Δx increments. The distribution $p(u_k)$ vs. u_k is then obtained by the normal graphical convolution mathematical calculations[Bracewell 1965, Healy 1969].

Figure 7


START
U = XY
Read
$p_X(x_i)$, $[x_i]_D$, Δx
$p_Y(y_k)$, y_k, u_j
$i = 1,n$
$j = 1,m$
$k = 1,b$
Compute
$[x_i]_D^{-1}$
j=1
Compute
$[y'_z] = u_j [x_i]_D^{-1}$
z=1
k=1
k=k+1
No
$y'_z > y_b$?
>
$y'_z : y_k$
=
$p_Y(y'_z)=p_Y(y_k)$
Yes
<
$p_Y(y'_z)=0$
Yes
$y'_z < y_1$?
No
Compute
$p_Y(y'_z)$ by
Interpolation
a
b
c
d
e

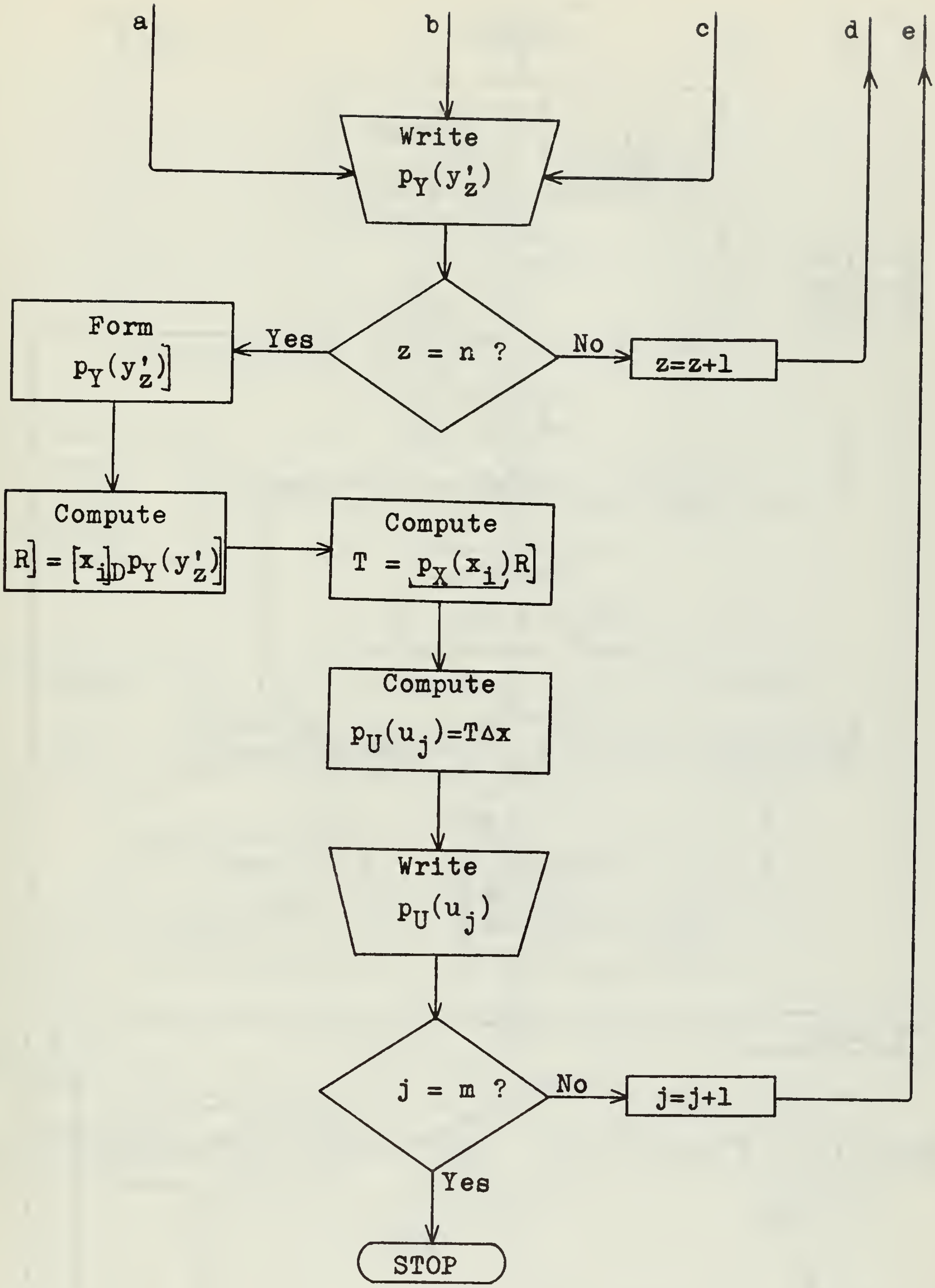
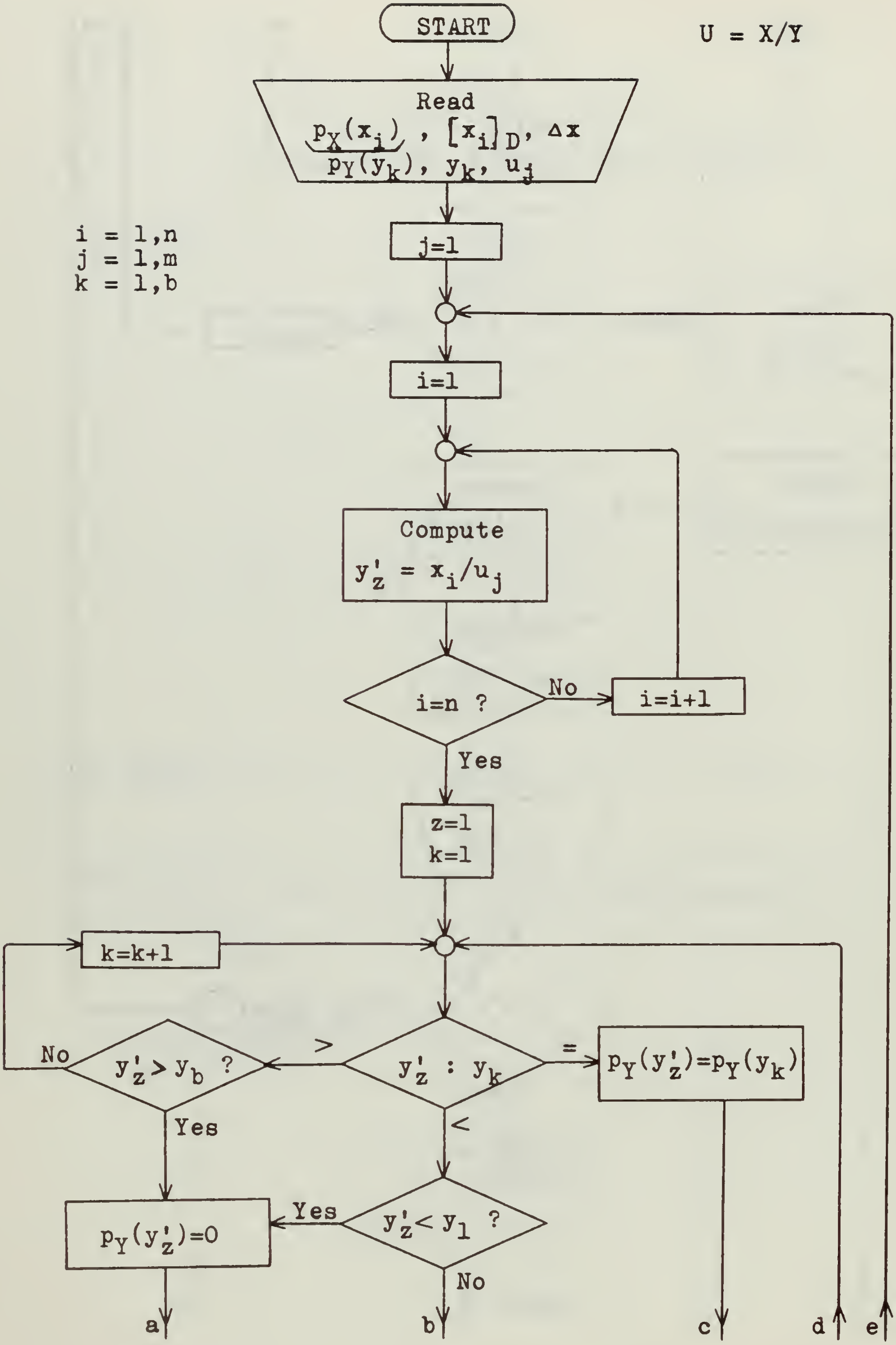

a
b
c
d
e
Write
$p_Y(y'_z)$
$z = n$?
Yes
No
$z=z+1$
Form
$p_Y(y'_z)]$
Compute
$R] = [x_i]_D p_Y(y'_z)]$
Compute
$T = \underbrace{p_X(x_i)}R]$
Compute
$p_U(u_j)=T\Delta x$
Write
$p_U(u_j)$
$j = m$?
No
$j=j+1$
Yes
STOP

Figure 8


START
$U = X/Y$
Read
$\underbrace{p_X(x_i)}$, $[x_i]_D$, Δx
$p_Y(y_k)$, y_k, u_j
$i = 1,n$
$j = 1,m$
$k = 1,b$
j=1
i=1
Compute
$y'_z = x_i/u_j$
i=n ?
No
i=i+1
Yes
z=1
k=1
k=k+1
No
$y'_z > y_b$?
>
$y'_z : y_k$
=
$p_Y(y'_z)=p_Y(y_k)$
Yes
<
$p_Y(y'_z)=0$
Yes
$y'_z < y_1$?
No
a
b
c
d
e

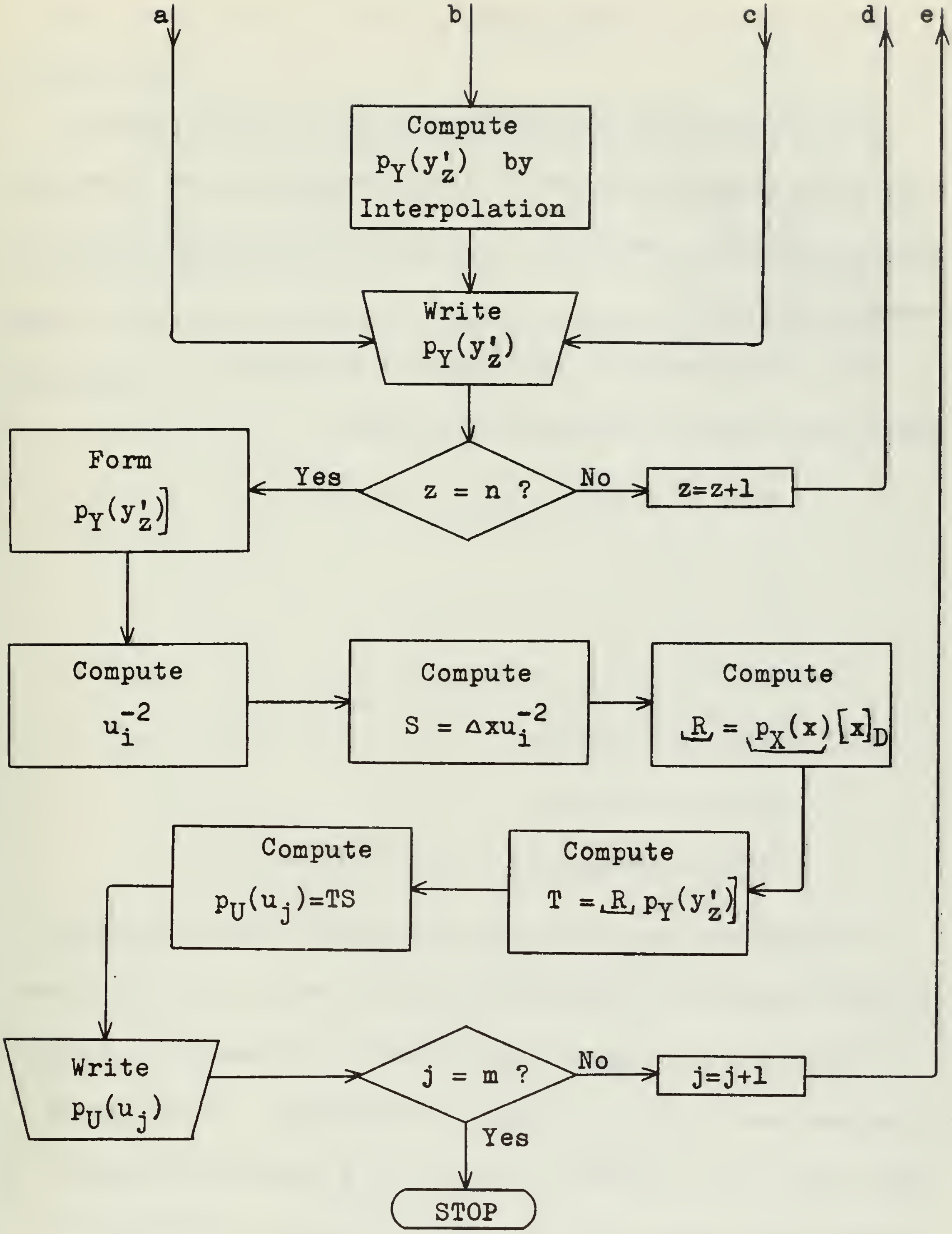
a
b
c
d
e
Compute $p_Y(y'_z)$ by Interpolation
Write $p_Y(y'_z)$
Form $p_Y(y'_z)]$
Yes
$z = n$?
No
$z=z+1$
Compute u_i^{-2}
Compute $S = \Delta x u_i^{-2}$
Compute $\underbrace{R} = \underbrace{p_X(x)}[x]_D$
Compute $p_U(u_j)=TS$
Compute $T = \underbrace{R}\, p_Y(y'_z)]$
Write $p_U(u_j)$
$j = m$?
No
$j=j+1$
Yes
STOP

Figure 9

APPENDIX I

The solution to the deterministic problem has been obtained by using three separate programs. Linear interpolation and rectangular approximations have been used to perform all convolution and rectangular integration.

The problem and given information is as follows:

Obtain the cumulative distribution of θ where

$$\theta = C + \frac{A}{X(x)} + \frac{C}{Y(y)} + W(w)$$

$C = 0.175$

$A = 8.1$

$B = 27.8$

$X(x)$ is $N(130;10)$

$Y(y)$ is $N(65;12)$

$W(w)$ is uniform 0.875 from $-4/7$ to $+4/7$

Program one inverts the distribution x vs. p(x) and multiplies it by the constant A. Program two does the same for $B/Y(y)$. Since both $X(x)$ and $Y(y)$ are normal distributions, the input data for these programs was taken from standard normal tables. 25 data points were used in each, the median and each 1/4 standard deviation out to 3 standard deviations. The resulting distributions, $X'(x)$ and $Y'(y)$, are then used as input data for program three. Program three performs the convolution $U(u) = X'(x) + Y'(y)$, the addition $U'(u) = U(u) + C$, and the convolution $\theta = U'(u) + W(w)$ to obtain the cumulative distribution of θ.

To obtain highly accurate results a Δx of 0.002 was used for convolution and rectangular integration. The over-all accuracy for this type of solution will depend upon how closely the rectangular approximations resemble the continuous distributions which they represent.

APPENDIX I - PROGRAM ONE

```
C        EVALUATION OF THE PROBABILITY DENSITY FUNCTION, P(X'),
C        WHERE X'=A/X AND A, X, AND P(X) ARE KNOWN. P(X) HAS
C        NORMAL DISTRIBUTION. N=NUMBER OF POINTS X(I), M IS THE
C        STANDARD DEVIATION OF THE DISTRIBUTION P(X).

         DIMENSION X(50),PX(50),U(50),PU(50)
         READ(5,100)N
         READ(5,101)M
         READ(5,102)A
     100 FORMAT(I2)
     101 FORMAT(I2)
     102 FORMAT(F10.0)
         WRITE(6,210) N
     210 FORMAT(1H1,T20,'N = ',I2//)
         WRITE(6,211) M
     211 FORMAT(T20,'M = ',I2//)
         WRITE(6,212) A
     212 FORMAT(T20,'A = ',F10.5//)
         WRITE(6,213)
     213 FORMAT(T20,'X(I)',T50,'PX(I)'//)
         DO 300 I=1,N
         READ(5,103) X(I),PX(I)
     103 FORMAT(2F10.0)
         WRITE(6,214) X(I),PX(I)
     214 FORMAT(2E30.8)
     300 CONTINUE
         DO 301 I=1,N
         K=N-I+1
         PU(K)=(X(I)**2/A)*(PX(I)/M)
         U(K)=A/X(I)
     301 CONTINUE
         WRITE(6,200)
     200 FORMAT(//,T20,'X',T50,'P(X )',//)
         DC 302 I=1,N
     201 FORMAT(2E30.8)
         STOP
         END

         N = 25
         M = 10
         A = 8.1

                       X'                                   PX'

                  0.50624996E-01                      0.13905164E 01
                  0.51428568E-01                      0.27868738E 01
                  0.52258059E-01                      0.51905861E 01
                  0.53114749E-01                      0.91015186E 01
                  0.53999994E-01                      0.14999995E 02
                  0.54915249E-01                      0.23179779E 02
                  0.55862062E-01                      0.33614029E 02
                  0.56842100E-01                      0.45776794E 02
                  0.57857137E-01                      0.53559014E 02
                  0.58909085E-01                      0.70279877E 02
                  0.59999995E-01                      0.79222473E 02
                  0.61132070E-01                      0.83914819E 02
                  0.62307686E-01                      0.83227264E 02
                  0.63529372E-01                      0.77608536E 02
                  0.64799964E-01                      0.67920517E 02
                  0.66122413E-01                      0.55782471E 02
                  0.67499995E-01                      0.43022202E 02
                  0.68936110E-01                      0.31123703E 02
                  0.70434749E-01                      0.21143661E 02
                  0.71999967E-01                      0.13484363E 02
                  0.73636353E-01                      0.80666637E 01
                  0.75348794E-01                      0.45226297E 01
                  0.77142835E-01                      0.23819437E 01
                  0.79024374E-01                      0.11803312E 01
                  0.80999970E-01                      0.54320967E 00
```

APPENDIX I - PROGRAM TWO

```
C      EVALUATION OF THE PROBABILITY DENSITY FUNCTION, P(Y'),
C      WHERE Y'=B/Y AND B, Y, AND P(Y) ARE KNOWN. P(Y) HAS
C      NORMAL DISTRIBUTION. N=NUMBER OF POINTS Y(I), M IS THE
C      STANDARD DEVIATION OF THE DISTRIBUTION P(Y).

       DIMENSION X(50),PX(50),U(50),PU(50)
       READ(5,100)N
       READ(5,101)M
       READ(5,102)A
   100 FORMAT(I2)
   101 FORMAT(I2)
   102 FORMAT(F10.0)
       WRITE(6,210) N
   210 FORMAT(1H1,T20,'N = ',I2//)
       WRITE(6,211) M
   211 FORMAT(T20,'M = ',I2//)
       WRITE(6,212) A
   212 FORMAT(T20,'A = ',F10.5//)
       WRITE(6,213)
   213 FORMAT(T20,'Y(I)',T50,'PY(I)'//)
       DO 300 I=1,N
       READ(5,103) X(I),PX(I)
   103 FORMAT(2F10.0)
       WRITE(6,214) X(I),PX(I)
   214 FORMAT(2E30.8)
   300 CONTINUE
       DO 301 I=1,N
       K=N-I+1
       PU(K)=(X(I)**2/A)*(PX(I)/M)
       U(K)=A/X(I)
   301 CONTINUE
       WRITE(6,200)
   200 FORMAT(//,T20,'Y',T50,'P(Y )',//)
       DO 302 I=1,N
   201 FORMAT(2E30.8)
       STOP
       END
```

```
       N = 25
       M = 12
       A = 27.8

                      Y'                                PY'

               0.27524740E 00                   0.13454551E 00
               0.28367329E 00                   0.26197964E 00
               0.29263145E 00                   0.47343379E 00
               0.30217373E 00                   0.80428284E 00
               0.31235939E 00                   0.12821751E 01
               0.32325566E 00                   0.19132919E 01
               0.33493960E 00                   0.26742373E 01
               0.34749985E 00                   0.35031176E 01
               0.36103874E 00                   0.43010139E 01
               0.37567550E 00                   0.49425173E 01
               0.39154911E 00                   0.53205528E 01
               0.40882331E 00                   0.53600159E 01
               0.42769212E 00                   0.50520163E 01
               0.44838685E 00                   0.44558611E 01
               0.47118622E 00                   0.36740417E 01
               0.49642831E 00                   0.28304844E 01
               0.52452803E 00                   0.20377035E 01
               0.55599976E 00                   0.13684053E 01
               0.59148908E 00                   0.85751051E 00
               0.63181788E 00                   0.50082946E 00
               0.67804843E 00                   0.27210426E 00
               0.73157859E 00                   0.13721460E 00
               0.79428536E 00                   0.64261079E-01
               0.86874962E 00                   0.27932845E-01
               0.95862025E 00                   0.11092324E-01
```

APPENDIX I - PROGRAM THREE

```
      CALCULATION OF U=X+Y WHERE X, P(X), Y, P(Y), ARE GIVEN
      AND THE GIVEN VALUES OF X(I) AND Y(I)  ARE INCREMENTED
      BY DX.

      DIMENSION X(500),PX(500),Y(500),PY(500),U(500),PU(500)
      PW(600),Z(1000),PZ(1000),F(500),W(600)
      READ(5,101) DX
  101 FORMAT(F10.0)
      WRITE(6,210) DX
  210 FORMAT(1H1,T20,'DX =',F10.3//)
      READ(5,106) C
  106 FORMAT(F10.0)
      WRITE(6,213) C
  213 FORMAT(T20,'C =',F10.8//)
      READ(5,105) N
  105 FORMAT(I5)
      WRITE(6,211) N
  211 FORMAT(T20,'N =',I5//)
      READ(5,104) M
  104 FORMAT(I5)
      WRITE(6,212) M
  212 FORMAT(//,T20,'M =',I5//)
      WRITE(6,202)
  202 FORMAT(//,T20,'X(I)',T50,'PX(I)'//)
      CALL ONE(X,PX)
      WRITE(6,203)
  203 FORMAT(//,T20,'Y(I)',T50,'PY(I)'//)
      CALL ONE(Y,PY)
      K=N+M-1
      U(1)=X(1)+Y(1)
      DO 302 I=2,K
  302 U(I)=U(I-1)+DX
      DO 303 J=1,K
      PU(J)=0.0
      DO 304 I=1,J
      IF((J-I+1).GT.M) GO TO 304
      IF(I.GT.N) GO TO 303
      PU(J)=PU(J)+PX(I)*PY(J-I+1)*DX
  304 CONTINUE
  303 CONTINUE

C     CALCULATION OF U(J)=U(J)+C WHERE 'C' IS A CONSTANT

      DO 306 I=1,K
      U(I)=U(I)+C
  306 CONTINUE
      WRITE(6,200)
  200 FORMAT(1H1,T20,'U(I)',T50,'PU(I)'//)
      DO 305 I=1,K
  305 WRITE(6,201) U(I),PU(I)
  201 FORMAT(2E30.8)

C     CALCULATION AND PLOT OF CUMULATIVE DISTRIBUTION

      F(1)=PU(1)*DX
      DO 320 I=2,K
      F(I)=F(I-1)+PU(I)*DX
  320 CONTINUE
      WRITE(6,225)
  225 FORMAT(1H1,T20,'U(J)',T40,'CUMULATIVE DISTRIBUTION')
      DO 321 I=1,K
      WRITE(6,201) U(I),F(I)
  321 CONTINUE

      PERFORM THE CONVOLUTION OF 'U' WITH THE CONSTANT PDF
      'W' (Z=U+W).

      CALL TWO(W,PW,L)
      NM=K+L-1
      Z(1)=U(1)+W(1)
      DO 310 I=2,NM
  310 Z(I)=Z(I-1)+DX
      DO 311 J=1,NM
      PZ(J)=0.0
      DO 312 I=1,J
      IF((J-I+1).GT.L) GO TO 312
      IF(I.GT.K) GO TO 311
      PZ(J)=PZ(J)+PU(I)*PW(J-I+1)*DX
  312 CONTINUE
  311 CONTINUE
      WRITE(6,230)
  230 FORMAT(1H1,T20,'Z(I)',T50,'PZ(I)'//)
      DO 313 I=1,NM
  313 WRITE(6,231) Z(I),PZ(I)
  231 FORMAT(2E30.8)

C     CALCULATION OF THE CUMULATIVE DISTRIBUTION OF 'Z'.

      PZ(1)=PZ(1)*DX
      DO 330 I=2,NM
      PZ(I)=PZ(I-1)+PZ(I)*DX
  330 CONTINUE
      WRITE(6,220)
  220 FORMAT(1H1,T20,'U(I)',T40,'CUMULATIVE DISTRIBUTION')
      DO 331 I=1,NM
  221 FORMAT(2E30.8)
  331 CONTINUE
      STOP
      END


      SUBROUTINE ONE(XE,PXE)

C     FROM GIVEN VALUES OF X(I) AND PX(I),(THESE ARE U(I)
      AND PU(I) PREVIOUSLY CALCULATED), AND A CHOSEN DX, WE
      OBTAIN THE VALUES OF X(J), WITH INTEVALS DX, AND
      CALCULATE THE CORRESPONDING VALUES OF PX(J) BY LINEAR
      INTERPOLATION.

      DIMENSION X(500),PX(500),XE(500),PXE(500)
      READ(5,100) N
  100 FORMAT(I2)
      READ(5,101) DX
  101 FORMAT(F10.0)
      DO 300 I=1,N
  300 READ(5,102) X(I), PX(I)
  102 FORMAT(2F10.0)
      I=1
      J=1
      XE(J)=X(I)+DX/2
  400 IF(XE(J)-X(I)) 500,501,502
  502 I=I+1
      IF(I.LT.(N+1)) GO TO 400
      GO TO 203
  501 PXE(J)=PX(I)
      GO TO 200
  500 PXE(J)=PX(I-1)+(PX(I)-PX(I-1))/(X(I)-X(I-1))*(XE(J)-
      X(I-1))
  200 WRITE(6,201)  XE(J), PXE(J)
  201 FORMAT(2E30.8)
      XE(J+1)=XE(J)+DX
      J=J+1
      GO TO 400
  203 RETURN
      END


      SUBROUTINE TWO(W,PW,L)

C     THIS PROGRAM TAKES THE GIVEN CONSTANT PROBABILITY
C     FUNCTION 'W', WITH AMPLITUDE 0.875 FROM -4/7 TO +4/7,
      AND OBTAINS THE VALUES OF W(I) WITH INCREMENT DX=0.002

      DIMENSION W(600), PW(600)
      W(1)=-0.571
      PW(1)=0.875
      DX=0.002
      L=571
      WRITE(6,250)
  250 FORMAT(1H1,T20,'W(I)',T50,'PW(I)'//)
      DO 350 I=2,L
      W(I)=W(I-1)+DX
      PW(I)=PW(I-1)
  350 CONTINUE
      DO 351 I=1,L
      WRITE(6,251) W(I),PW(I)
  251 FORMAT(2E30.8)
  351 CONTINUE
      RETURN
      END


      N  = 15
      M  = 242
      C  = 0.175
      DX = 0.002
```

```
        THETA                                    CUMULATIVE
                                                DISTRIBUTION

   -0.68128109E-01                            0.35162280E-08
   -0.66128075E-01                            0.24354559E-07
   -0.64128041E-01                            0.93342241E-07
   -0.62128045E-01                            0.29366913E-06
   -0.60128044E-01                            0.71189930E-06
   -0.58128044E-01                            0.14791967E-05
   -0.56128044E-01                            0.27345241E-05
   -0.54128043E-01                            0.46203268E-05
   -0.52128043E-01                            0.72799021E-05
   -0.50128043E-01                            0.10858982E-04
   -0.48128042E-01                            0.15508529E-04
   -0.46128042E-01                            0.21388027E-04
   -0.44128042E-01                            0.28669005E-04
   -0.42128041E-01                            0.37537306E-04
   -0.40128041E-01                            0.48194212E-04
   -0.38128041E-01                            0.60856095E-04
   -0.36128040E-01                            0.75755495E-04
   -0.34128040E-01                            0.93142065E-04
   -0.32128040E-01                            0.11328337E-03
   -0.30128039E-01                            0.13646450E-03
   -0.28128039E-01                            0.16298697E-03
   -0.26128039E-01                            0.19316789E-03
   -0.24128038E-01                            0.22734021E-03
   -0.22128038E-01                            0.26585301E-03
   -0.20128038E-01                            0.30907127E-03
   -0.18128037E-01                            0.35737362E-03
   -0.16128037E-01                            0.41115121E-03
   -0.14128037E-01                            0.47080661E-03
   -0.12128036E-01                            0.53675333E-03
   -0.10128036E-01                            0.60941558E-03
   -0.81280358E-02                            0.68922667E-03
   -0.61280355E-02                            0.77662780E-03
   -0.41280352E-02                            0.87206601E-03
   -0.21280353E-02                            0.97599253E-03
   -0.12803543E-03                            0.10888616E-02
    0.18719644E-02                            0.12111294E-02
    0.38719643E-02                            0.13432535E-02
    0.58719628E-02                            0.14856923E-02
    0.78719594E-02                            0.16389030E-02
    0.98719560E-02                            0.18033390E-02
    0.11871953E-01                            0.19794477E-02
    0.13871949E-01                            0.21676687E-02
    0.15871946E-01                            0.23684341E-02
    0.17871942E-01                            0.25821684E-02
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0.17758036E 01                              0.99590898E 00
0.17778034E 01                              0.99590898E 00
0.17798033E 01                              0.99590898E 00
0.17818031E 01                              0.99590898E 00
```

APPENDIX II

The tabulated sample data for the non-deterministic case, Table II, was obtained from a modified form of equation (19). Using the modified equation,

$$\phi = 0.175 + \frac{27.8}{Y(y)} + \frac{8.1}{X(x)}$$

where $X(x)$ is $N(130;10)$

$Y(y)$ is $N(65;12)$

values of ϕ and sensitivity were calculated for chosen values of x and y. Assuming these to be measured values, the data was then used to obtain the cumulative distribution of ϕ.

Since the data indicates ϕ is more sensitive to changes in y than to changes in x, the problem is worked with x as a constant and y as a single variable as discussed in section III. Aitkin-Hermite interpolation [Hildebrand 1956] is used to estimate the values of y between data points. This interpolation scheme utilizes the sensitivity data and obtains a more accurate estimate than would linear interpolation. The calculation of the cumulative distribution of ϕ is then made using equation (35) in discrete form. As in the deterministic case, a Δx of 0.002 was used for rectangular integration.

APPENDIX II - PROGRAM ONE

```
C     CALCULATION OF THE CUMULATIVE DISTRIBUTION OF AN
      OUTPUT,  PHI  , FROM MEASURED VALUES OF THE PARAMETERS
      X, Y, AND THE SENSITIVITY OF  PHI   WITH RESPECT TO X
      AND Y. THE PROBABILITY DISTRIBUTIONS OF X AND Y ARE
      KNOWN, N(130;10) AND N(65;12) RESPECTIVELY, AND THE
      DATA POINTS HAVE BEEN SELECTED IN DX INTERVALS TO
      FACILITATE THE CALCULATIONS.

      DIMENSION ARG1(5,5),VAL(10),ARG(5),XC(5),F(5,2),Z(5),
      PYT(25),PXC(5),X1(200),PX(200),YT(25)

C     READ THE MEASURED VALUES OF X AND THEIR CORRESPONDING
C     PROBABILITIES, DENOTED BY XC AND PXC.

      READ(5,102) (XC(I),PXC(I),I=1,5)
      WRITE(6,103)
  103 FORMAT(//,T21,'XC',T47,'PXC',/)
      WRITE(6,106) (XC(I),PXC(I),I=1,5)
  106 FORMAT(2F26.5)
  101 FORMAT(F10.5)

      READ THE MEASURED VALUES OF  PHI  , DENOTED BY ARG1.
      THERE IS A POINT  PHI   CORRESPONDING TO EACH SET OF
      X - Y POINTS.

      READ(5,101) ((ARG1(I,J),J=1,5),I=1,5)

C     READ THE MEASURED VALUES OF Y AND THE CORRESPONDING
      SENSITIVITY OF  PHI   WITH RESPECT TO Y FOR EACH OF
      THESE VALUES, DENOTED BY F(I,J).

      READ(5,102) ((F(I,J),J=1,2),I=1,5)
  102 FORMAT(2F10.5)

      READ THE KNOWN CUMULATIVE DISTRIBUTION OF Y, DENOTED
      BY YT AND PYT.

      READ(5,102) (YT(I),PYT(I),I=1,25)
      X1(1)=0.5708710
      DO 204 K=1,168
      PT=0.0

C     X DENOTES THE VALUE OF  PHI   FOR WHICH THE CUMULATIVE
C     DISTRIBUTION IS BEING CALCULATED.

      X=X1(K)
      DO 202 I=1,5
      DO 201 J=1,5

      Z(J) DENOTES THE ARRAY OF ARG1 VALUES CORRESPONDING
      TO A PARTICULAR VALUE OF XC.

  201 Z(J)=ARG1(I,J)
      DX=10.0
      EPS=0.001
      NDIM=5

C     SUBROUTINE ATSM ORDERS THE ARRAY Z(J) SUCH THAT
      ABS(ARG(I)-X).GE.ABS(ARG(J)-X) IF I.GEJ. THIS IS
      REQUIRED FOR SUBROUTINE ONE.

      CALL ATSM(X,Z,F,5,2,ARG,VAL,NDIM)

C     SUBROUTINE ONE INTERPOLATES FUNCTION VALUE Y FOR A
      GIVEN ARGUMENT VALUE X USING THE TABLE (ARG,VAL) OF
      ARGUMENT, FUNCTION, AND DERIVATIVE VALUES FROM THE
C     SUBROUTINE ATSM. THIS SUBROUTINE USES THE AITKEN-
      HERMITE INTERPOLATION.
      CALL ONE(X,ARG,VAL,Y,NDIM,EPS,IER)

C     SUBROUTINE TWO USES LINEAR INTERPOLATION TO OBTAIN
      P(Y) FOR THE VALUE OF Y OBTAINED IN SUBROUTINE ONE.

      CALL TWO(Y,PY,YT,PYT)
      PT=PT+(1-PY)*PXC(I)
      PX(K)=PT*DX
  302 FORMAT(2E30.8)
  202 CONTINUE
      X1(K+1)=X1(K)+0.0020000
  204 CONTINUE
      WRITE(6,107)
  107 FORMAT(1H1,T20,'U(I)',T42,'CUMULATIVE DISTRIBUTION',//
      WRITE(6,302) (X1(K),PX(K),K=1,168)
      STOP
      END


      SUBROUTINE ATSM(X,Z,F,IROW,ICOL,ARG,VAL,NDIM)

      DIMENSION Z(1),F(1),ARG(1),VAL(1)

C     CASE IROW=1 IS CHECKED OUT
      IF(IROW-1)23,21,1
    1 N=NDIM

C     IF N IS GREATER THAN IROW, N IS SET EQUAL TO IROW.
      IF(N-IROW)3,3,2
    2 N=IROW

C     CASE IROW .GE. 2
C     SEARCHING FOR SUBSCRIPT J SUCH THAT Z(J) IS NEXT TO X.
    3 IF(Z(IROW)-Z(1))5,4,4
    4 J=IROW
      I=1
      GO TO 6
    5 I=IROW
      J=1
    6 K=(J+I)/2
      IF(X-Z(K))7,7,8
    7 J=K
      GO TO 9
    8 I=K
    9 IF(IABS(J-I)-1)10,10,6
   10 IF(ABS(Z(J)-X)-ABS(Z(I)-X))12,12,11
   11 J=I

C     TABLE SELECTION
   12 K=J
      JL=0
      JR=0
      DO 20 I=1,N
      ARG(I)=Z(K)
      IF(ICOL-1)14,14,13
   13 VAL(2*I-1)=F(K)
      KK=K+IROW
      VAL(2*I)=F(KK)
      GO TO 15
   14 VAL(I)=F(K)
   15 JJR=J+JR
      IF(JJR-IROW)16,18,18
   16 JJL=J-JL
      IF(JJL-1)19,19,17
   17 IF(ABS(Z(JJR+1)-X)-ABS(Z(JJL-1)-X))19,19,18
   18 JL=JL+1
      K=J-JL
      GO TO 20
   19 JR=JR+1
      K=J+JR
   20 CONTINUE
      RETURN
```

```
C     CASE IFOW=1
   21 ARG(1)=Z(1)
      VAL(1)=F(1)
      IF(ICOL-2)23,22,23
   22 VAL(2)=F(2)
   23 RETURN
      END

      SUBROUTINE ONE(X,ARG,VAL,Y,NDIM,EPS,IER)

      DIMENSION ARG(1),VAL(1)
      IER=2
      H2=X-ARG(1)
      IF(NDIM-1)2,1,3
    1 Y=VAL(1)+VAL(2)*H2
    2 RETURN
C
C     VECTOR ARG HAS MORE THAN 1 ELEMENT.
C     THE FIRST STEP PREPARES VECTOR VAL SUCH THAT AITKEN
      SCHEME CAN BE USED.
    3 I=1
      DO 5 J=2,NDIM
      H1=H2
      H2=X-ARG(J)
      Y=VAL(I)
      VAL(I)=Y+VAL(I+1)*H1
      H=H1-H2
      IF(H)4,13,4
    4 VAL(I+1)=Y+(VAL(I+2)-Y)*H1/H
    5 I=I+2
      VAL(I)=VAL(I)+VAL(I+1)*H2
C     END OF FIRST STEP

C     PREPARE AITKEN SCHEME
      DELT2=0.
      IEND=I-1

C     START AITKEN-LOOP
      DO 9 I=1,IEND
      DELT1=DELT2
      Y=VAL(1)
      M=(I+3)/2
      H1=ARG(M)
      DO 6 J=1,I
      K=I+1-J
      L=(K+1)/2
      H=ARG(L)-H1
      IF(H)6,14,6
    6 VAL(K)=(VAL(K)*(X-H1)-VAL(K+1)*(X-ARG(L)))/H
      DELT2=ABS(Y-VAL(1))
      IF(DELT2-EPS)11,11,7
    7 IF(I-5)9,8,8
    8 IF(DELT2-DELT1)9,12,12
    9 CONTINUE
C     END OF AITKEN-LOOP

   10 Y=VAL(1)
      RETURN

      THERE IS SUFFICIENT ACCURACY WITHIN 2*NDIM-2 ITERATION
      STEPS
   11 IER=0
      GOTO 10

C     TEST VALUE DELT2 STARTS OSCILLATING
   12 IER=1
      RETURN

C     THERE ARE TWO IDENTICAL ARGUMENT VALUES IN VECTOR ARG
   13 Y=VAL(1)
   14 IER=3
```

```
      RETURN
      END

      SUBROUTINE TWO(Y,PY,YT,PYT)

      DIMENSION YT(25),PYT(25)
      IF(Y.LT.YT(1)) GO TO 301
      J=2
  110 IF(Y-YT(J)) 200,201,202
  202 J=J+1
      IF(J.LE.25) GO TO 110
      PY=1.0
      GO TO 300
  201 PY=PYT(J)
      GO TO 300
  200 PY=PYT(J-1)+(PYT(J)-PYT(J-1))/(YT(J)-YT(J-1))*(Y-YT(J-
      GO TO 300
  301 PY=0.0
  300 RETURN
      END
```

```
                                                CUMULATIVE
         PHI                                   DISTRIBUTION

0.57087100E 00                               0.62946975E-01
0.57287097E 00                               0.68146110E-01
0.57487094E 00                               0.74001014E-01
0.57687092E 00                               0.79798685E-01
0.57887089E 00                               0.85878432E-01
0.58087087E 00                               0.92248380E-01
0.58287084E 00                               0.98956390E-01
0.58487082E 00                               0.10555303E 00
0.58687079E 00                               0.11298674E 00
0.58887076E 00                               0.12032640E 00
0.59087074E 00                               0.12787700E 00
0.59287071E 00                               0.13566995E 00
0.59487069E 00                               0.14363253E 00
0.59687066E 00                               0.15177727E 00
0.59887064E 00                               0.16019422E 00
0.60087061E 00                               0.16903991E 00
0.60287058E 00                               0.17780799E 00
0.60487056E 00                               0.18684916E 00
0.60687053E 00                               0.19600827E 00
0.60887051E 00                               0.20516866E 00
0.61087048E 00                               0.21465921E 00
0.61287045E 00                               0.22408015E 00
0.61487043E 00                               0.23413759E 00
0.61687040E 00                               0.24408901E 00
0.61887038E 00                               0.25411922E 00
0.62087035E 00                               0.26433289E 00
0.62287033E 00                               0.27449834E 00
0.62487030E 00                               0.28468227E 00
0.62687027E 00                               0.29506922E 00
0.62887025E 00                               0.30539531E 00
0.63087022E 00                               0.31610876E 00
0.63287020E 00                               0.32671946E 00
0.63487017E 00                               0.33738047E 00
0.63687015E 00                               0.34810561E 00
0.63887012E 00                               0.35874993E 00
0.64087009E 00                               0.36933064E 00
0.64287007E 00                               0.37996292E 00
0.64487004E 00                               0.39054102E 00
0.64687002E 00                               0.40114039E 00
0.64886999E 00                               0.41177332E 00
0.65086997E 00                               0.42229414E 00
0.65286994E 00                               0.43286431E 00
0.65486991E 00                               0.44332552E 00
0.65686989E 00                               0.45370537E 00
0.65886986E 00                               0.46397537E 00
0.66086984E 00                               0.47415489E 00
0.66286981E 00                               0.48424327E 00
0.66486979E 00                               0.49421632E 00
0.66686976E 00                               0.50409496E 00
0.66886973E 00                               0.51387882E 00
0.67086971E 00                               0.52356750E 00
0.67286968E 00                               0.53316504E 00
0.67486966E 00                               0.54267335E 00
0.67686963E 00                               0.55209064E 00
0.67886961E 00                               0.56139952E 00
0.68086958E 00                               0.57061529E 00
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0.68486953E 00                               0.58854502E 00
0.68686950E 00                               0.59722644E 00
0.68886948E 00                               0.60576999E 00
0.69086945E 00                               0.61422157E 00
0.69286942E 00                               0.62248212E 00
0.69486940E 00                               0.63067019E 00
0.69686937E 00                               0.63879596E 00
0.69886935E 00                               0.64680159E 00
0.70086932E 00                               0.65472543E 00
0.70286930E 00                               0.66255569E 00
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0.70686924E 00                               0.67769872E 00
```

```
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```

```
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```

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13. ABSTRACT

Techniques for extrapolating statistical data on the parameters of individual components to the statistical performance index for an overall system are considered. Two cases are evaluated. (1) The deterministic case in which the system's performance index is known functionally in terms of the system parameters. (2) The non-deterministic case in which only limited data on the performance index and its sensitivity with respect to system parameters is known. Computer programs are developed in both cases for combining given probability density distributions for the parameters into an overall probability density distribution for the system performance index. Theory and programs are developed and verified with a specific numerical example.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Deterministic						
Jacobian						
Non-deterministic						
Probability Density						
Sensitivity Coefficient						

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